Correlation dimension of attractors through interspike intervals

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The question whether the correlation dimension of an experimental chaotic dynamical system can be determined from a series of spike timings produced by the system is considered. Two separate, biologically motivated methods for generating spikes from the system are surveyed: an integrate-and-fire model and a threshold-crossing method. Computational evidence indicates that the dimension can be determined in principle under either method, assuming that the embedding dimension of the interspike interval attractor reconstruction is greater than or equal to the correlation dimension of the attractor. The dependence of the dimension statistics on the threshold level and the amount of available data is investigated. [S1063-651X(97)11001-7]

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Grassberger and Procaccia [1] formulated a method for determining the correlation dimension of a chaotic attractor of a system using a time series observed from the system. The method is based on the idea of the geometric reconstruction of an attractor described earlier by Packard *et al.*. [2], Roux *et al.* [3], and others. Takens [4] proved a mathematical theorem about the preservation of topology and geometry under time series reconstructions. Widespread dissemination of these ideas led to use of nonlinear time series techniques in many areas of science.

According to [5], a delay coordinate reconstruction of a compact attractor from a time series is topologically equivalent to the attractor for a probability-1 choice of measurement functions, as long as the embedding dimension is greater than twice the box-counting dimension of the attractor. This dimension requirement is overkill if the goal is simply to measure the attractor dimension. In [6], it was proved that a delay coordinate reconstruction of a compact attractor from a time series has the same correlation dimension as the attractor for a probability-1 choice of measurement functions, assuming that the embedding dimension is greater than or equal to the correlation dimension of the attractor. Departure from this theory in practice should be traced to the lack of sufficient high-precision data or nongenericity of measurement functions.

In certain scientific areas, time series are not available for dynamical characterization. In some cases, data measured from a dynamical process are collected as interevent timings, either because that form is more convenient or more representative of the process. Biologists in general and neurophysiologists in particular often prefer to analyze spike trains, although the preference is not limited to these areas [7]. Recent laboratory experiments on control of irregular behavior in biological systems have led to concentrated work on the analysis of spike trains for the reliable detection of unstable periodic orbits in the underlying system [8].

This paper is concerned with the following question: If an attractor can be measured only from series of spike times, can its dimension be determined from the spike information alone? In the time series case, we quoted a result above that holds for time series resulting from generic measurement functions. Although it is not always clear in practice what a generic time series is for a given underlying process, at least there is a clear mathematical notion. It is not so obvious, even mathematically, what a generic spike train generated from a process should mean. For that reason, we will survey two different methods of generating spikes, and demonstrate that, under either one, the correct correlation dimension can be inferred from spike train measurements.

The first of the two methods is a general integrate-and-fire model. This method was shown in [9] to allow the full reconstruction of the dynamics using a delay embedding of interspike intervals (ISI's). A generic time series from a dynamical system is integrated with respect to time; when it reaches a preset threshold, a spike is generated, after which the integration is restarted. The second method is a threshold-crossing model. The times that a generic time series crosses a fixed threshold from below to above are recorded. Unlike the first method, time intervals between threshold upcrossings will not be sufficient to reconstruct the attractor. In certain cases they may reconstruct a rough analogy to a Poincaré section.

We begin by considering the integrate-and-fire hypothesis. Let $S(t)$ represent a signal produced by a function of the variables of a finite-dimensional dynamical system. Assume that the trajectories of the dynamical system are asymptotic to a compact attractor. Let Θ be a positive number which represents the firing threshold. After fixing a starting time T_0, a series of ''firing times'' $T_1<T_2<T_3<\cdots$ can be recursively defined by the equation

$$\int_{T_i}^{T_{i+1}} S(t)dt=\Theta. \tag{1}$$

From the firing times T_i, the interspike intervals can be defined as $t_i=T_i-T_{i-1}$. Figure 1 shows a trace of the x coordinate of the Lorenz attractor [10], governed by the equations

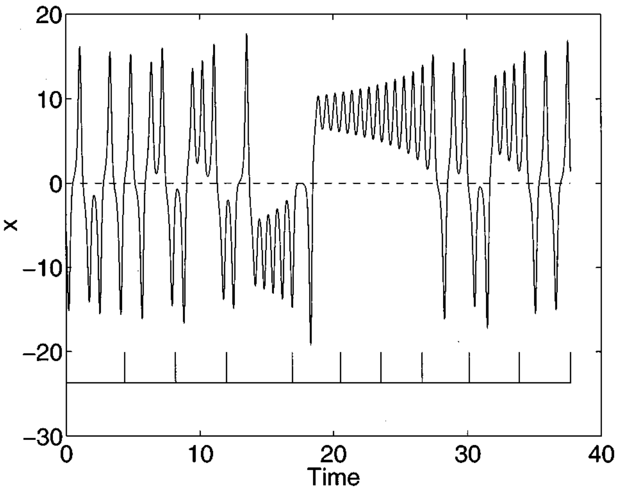

FIG. 1. The upper trace $x(t)$ is the x coordinate of the Lorenz attractor graphed as a function of time. The lower trace shows the times at which spikes are generated using $S(t)=x(t)+25$ and $\Theta=10$.

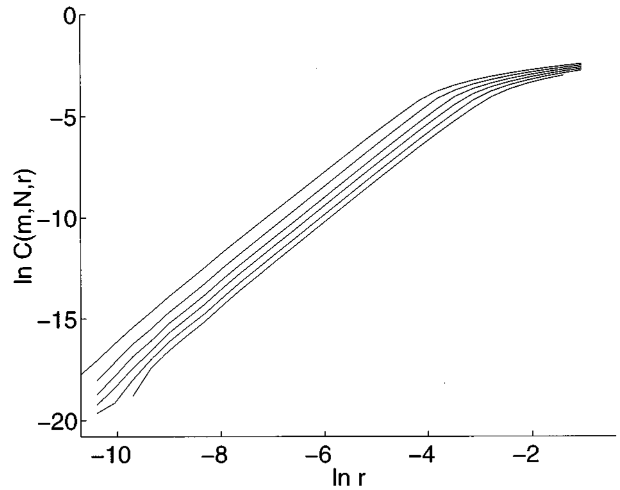

FIG. 2. Ln-ln plot of the correlation integrals of the Lorenz ISI [signal $S(t)=x(t)+25$, $\Theta=10$] for $m=3$–8. The top line is for $m=3$, the bottom line $m=8$.

$$\dot{x}=\alpha(y-x),$$

$$\dot{y}=\rho x-y-xz, \qquad (2)$$

$$\dot{z}=-\beta z+xz,$$

where the parameters are set to the standard values $\alpha=10$, $\rho=28$, and $\beta=8/3$, together with spiking times generated by Eq. (1) where $S(t)=x(t)+25$ and $\Theta=10$. This choice for $S(t)$ is representative. A wide range of positive functions $S(t)$ yielded results similar to those about to be described. The threshold Θ, on the other hand, turns out to be a critical parameter, and is discussed below in more detail.

The correlation dimension D_2 of an invariant measure of a dynamical system is given by

$$D_2=\lim_{r\to 0}\frac{\ln C(r)}{\ln r}, \qquad (3)$$

where the correlation integral $C(r)$ is defined to be the probability that a pair of points chosen at random is separated by a distance less than r. Let $\{\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_N\}$ be the first N points of a trajectory which traces out the invariant measure. The correlation integral can be approximated by

$$C(N,r)=\frac{2}{N(N-1)}\sum_{j=1}^{N}\sum_{i=j+1}^{N}H(r-|\mathbf{x}_i-\mathbf{x}_j|), \qquad (4)$$

where H is the Heaviside function defined as $H(x)=0$ for $x\leqslant 0$, and $H(x)=1$ for $x>0$, and $|\,|$ denotes the distance norm. As N goes to ∞, $C(N,r)$ goes to $C(r)$.

Given a sequence of ISI's we can construct an m-dimensional space using delay coordinates and compute $C(m,N,r)$ according to Eq. (4) for a given m. (Here a parameter m is added to indicate the dimensionality of the reconstruction space.) If the estimated values $\overline{D_2^{(m)}}$, plotted as a function of m, appears to reach a plateau for a range of large enough m, then the plateau dimension value is taken to be an estimate for D_2.

In Fig. 2 we plot $\ln C(m,N,r)$ versus $\ln r$ for $m=3$ to 8, and $N=64\,000$. The ISI's were generated using the x coordinate of the Lorenz attractor, the signal $S(t)=x(t)+25$ and $\Theta=10$. An optimized box-assisted algorithm for correlation dimension [11] was used to compute $C(m,N,r)$.

To estimate D_2, Grassberger and Procaccia [1] suggest that a ln-ln plot of $C(N,r)$ versus r be constructed and that the estimation of the dimension, denoted by $\overline{D_2}$, be read off as the slope of the curve over some range where the graph shows a linear dependence. A variant of this method was suggested by Takens [12]. The Takens estimator requires the choice of a single free parameter R_0 as the upper cutoff distance. All pairwise distances larger than R_0 are discarded, and all distances r which are less than R_0 are averaged according to

$$d=\frac{-1}{\langle \ln(r/R_0)\rangle}. \qquad (5)$$

This d is an estimate for D_2. We report results using the Takens estimator of dimension as a way of making our conclusions less sensitive to choice of scaling region in the determination of dimension, in particular for cases that are less clear cut than Fig. 2.

An important issue to be considered for ISI's from integrate-and fire dynamics is the effect of the size of the threshold in the reconstruction. As the threshold increases, the length of time spanned by an ISI vector grows, and decorrelation due to sensitive dependence can damage the reconstruction. To see this we compute the correlation dimension of ISI series generated with the Lorenz equations using different thresholds.

In Fig. 3, $\overline{D_2^{(m)}}$ is plotted versus m for $\Theta=10$ and 50. The signal and trajectory used are the same as in Fig. 2. We used the Takens estimator to compute $\overline{D_2^{(m)}}$ with R_0 equal to one-half of one percent of the diameter of the reconstructed at-

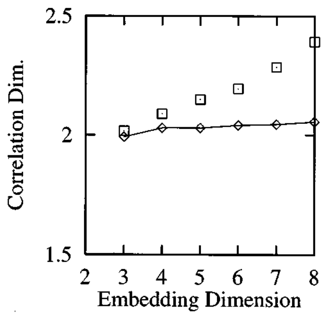

FIG. 3. Dimension estimates of integrate-and-fire Lorenz system measured from interspike intervals, as a function of embedding dimension m. Takens estimator with $R_0 = 0.5\,\%$ of diameter is used. The diamond symbols mark dimension estimator for threshold $\Theta = 10$; squares mark threshold $\Theta = 50$.

tractor. We were able to reconstruct the Lorenz attractor dimension using $\Theta = 10$, but not with $\Theta = 50$. As shown in the graph, with $\Theta = 10$ (diamonds), the plot reaches a plateau close to the Lorenz correlation dimension 2.06. This is not the case for $\Theta = 50$. Increasing the threshold by a factor of 5 results in an increase in the mean interspike interval by a factor of roughly 5. Referring to Fig. 1, where $\Theta = 10$, one concludes that the system moves through many oscillations during each ISI when $\Theta = 50$. Since the Lorenz attractor has a Lyapunov exponent of approximately 0.9, information about the original system state is lost after only one or two spike intervals. At the precision with which the ISI's were measured in this study, useful information about the attractor is not reproduced in the reconstruction. This problem for ISI's is exactly analogous to the effect of large time delays when reconstructing an attractor from time series.

The second method for generating a spike sequence is to record the time intervals between successive, upward-going level crossings of a fixed amplitude threshold Θ. Figure 4 shows the xy projection of the Rössler attractor [13], governed by the equations

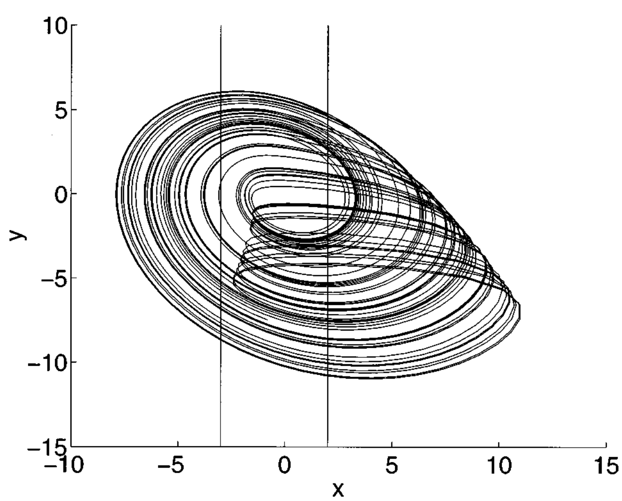

FIG. 4. An xy projection of the Rössler attractor. The lines $x=2$ and -3 represent possible firing thresholds Θ.

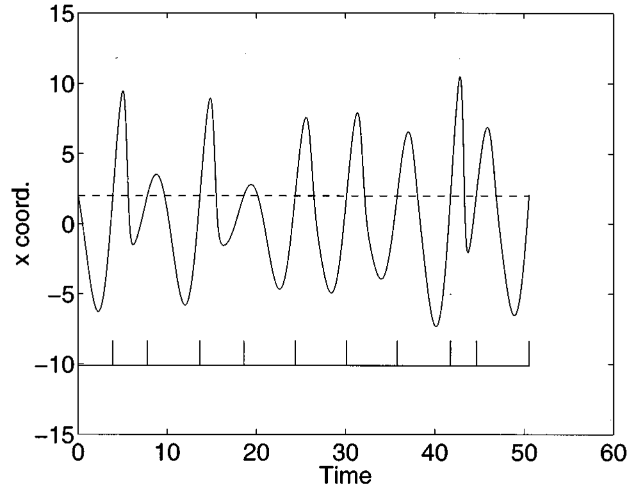

FIG. 5. The upper trace is the x coordinate of the Rössler attractor graphed as a function of time. The lower trace shows the times at which threshold crossings are generated using $\Theta = 2$.

$$\dot{x} = -(y+z),$$
$$\dot{y} = x + ay, \tag{6}$$
$$\dot{z} = b + (x-c)z,$$

where the parameters are set to the standard values $a = 0.36$, $b = 0.4$, $c = 4.5$, together with the lines $x=2$ and

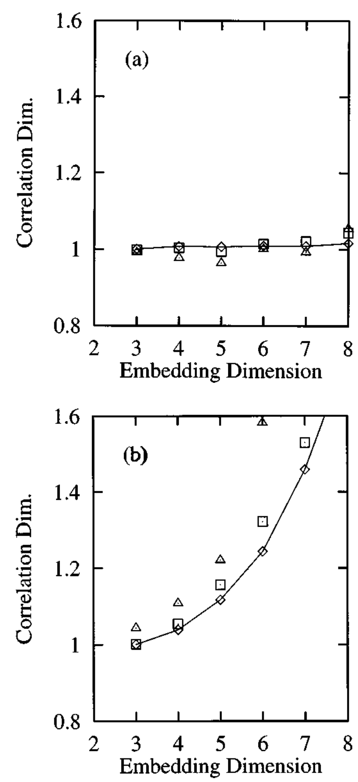

FIG. 6. Dimension estimates of Rössler system measured from threshold-crossing interspike intervals, using Takens estimator. The diamond, square, and triangle symbols correspond to ISI sequence lengths 64 000, 16 000, and 4000, respectively. The thresholds in the x variable are (a) $\Theta = 2$ and (b) $\Theta = -3$.

-3. Figure 5 shows a trace of the x coordinate of the Rössler attractor, together with the threshold crossings using $\Theta=2$.

Interspike intervals created by threshold crossings of a system measurement cannot be used to reconstruct the underlying attractor in the sense of [9]. Instead, these ISI's measure the times between piercings of a Poincaré surface of section. We conjecture that vectors consisting of m successive interspike intervals created in this way will generically comprise a set of dimension exactly one less than the attractor dimension.

Computational evidence of this conjecture is shown in Fig. 6(a), which shows dimension estimates, using the Takens estimator as above, for threshold-crossing ISI's with $\Theta=2$. As shown by Fig. 5, this threshold is chosen so that firing occurs with the great majority of the oscillations of the underlying attractor. The dimension estimates are close to 1; for the diamonds shown in Fig. 6(a), representing the dimension calculation using 64 000 interspike intervals, the dimension estimate is 1.01, essentially equal to one less than the correlation dimension of the Rössler attractor. When fewer than 64 000 intervals are used, the dimension estimate degrades in accuracy. The squares and triangles in Fig. 6(a) represent the dimension estimates for 16 000 and 4000 intervals, respectively.

The alternate threshold $\Theta=-3$ in Fig. 4 leads to quite a different result. As can be seen from either Fig. 4 or 5, many of the smaller oscillations will now be missed by the threshold-crossing method. Although our conjecture is that in theory the correct dimension can be found from these ISI's, in practice the data requirements are difficult to satisfy. The likely cause of the difference between thresholds $\Theta=2$ and -3 is that the latter produces fewer spikes per unit time, because small oscillations are missed. For a fixed embedding dimension, the length of dynamical time spanned by a reconstructed ISI vector becomes comparable with the decorrelation time of the chaotic system, which has the effect of degrading the attractor reconstruction. Figure 6(b) shows the resulting inaccuracy in the dimension estimate. This is perhaps consistent with the findings of Preissl, Aersten, and Palm [14], who reported difficulty in obtaining the correct correlation dimension of the Lorenz attractor from threshold crossings, possibly due to this decorrelation effect. In that study, 5000 ISI's were used in the dimension calculation.

Taken collectively, these results show that there is no theoretical obstruction to computing correlation dimension from a single spike train measured from experiment. Our methods for producing spikes from underlying dynamics were necessarily limited in scope. Real experimental systems are certain to be much more complex in detail, and possibly not well represented by either integrate-and-fire or threshold-crossing mechanisms. The contribution of this report is to show what happens under ideal circumstances, given sufficient accurately-measured data. We also showed that dimension calculations from ISI's in practice are highly dependent on the details of the firing threshold. In a typical experiment, lack of control over this parameter could conceivably make an accurate dimension estimation extremely difficult.

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